

Supersymmetric Gauge Theories on the Five-Sphere

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ABSTRACT: We construct Euclidean 5d supersymmetric gauge theories on the five-sphere with vector and hypermultiplets. The SUSY transformation and the action are explicitly determined from the standard Noether procedure as well as from off-shell supergravity. Using localization techniques, the path-integral is shown to be restricted to the integration over a generalization of instantons on \mathbb{CP}^2 and the Coulomb moduli.

KEYWORDS: Supersymmetric gauge theory.

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1. Introduction

Recently, supersymmetric (SUSY) gauge theories on curved spaces have been studied intensively. One of the reasons why such studies are important is that the partition function and some correlators are exactly computable by localization techniques. In general, however, it is not obvious whether one can construct SUSY gauge theories on a given curved space. Even if it is possible, one has to construct the theories one by one.

SUSY gauge theories on some simple curved spaces have caught attention for many years.¹ One of the simplest classes of SUSY gauge theories on curved spaces are on d -dimensional spheres S^d . SUSY gauge theories on S^4 were considered in [3], where it was shown that the partition function as well as the expectation values of certain Wilson loops can be computed as certain matrix integrals which were first conjectured by [4, 5]. The exact results also led to the discovery of a surprising relation between 4d $\mathcal{N} = 2$ SUSY gauge theories and 2d conformal field theories [6, 7]. For three dimensions, an exact computation was initiated by [8] and generalized by [9, 10].

Following these successes in finding SUSY theories on S^4 and S^3 , it is natural to extend the search and to consider SUSY gauge theories on S^5 . Although 5d gauge theories are not perturbatively renormalizable, one can consider any UV completion of the theory on S^5 . Then, if localization gives an exact result which is independent of how one completes the theory in the UV, the result may be well-defined.

Adding to this, there is another strong motivation to consider these theories. In [11, 12], it was proposed that the maximal SUSY 5d gauge theory describes the 6d $\mathcal{N} = (2, 0)$ SUSY conformal field theory compactified on a circle without introducing Kaluza-Klein degrees of freedom. This 6d $\mathcal{N} = (2, 0)$ CFT is both very interesting and mysterious. There is no intrinsic definition of

¹There have been constructions of SUSY gauge theories on curved spaces based on topological twisting [1, 2].

the theory, and we only know of it through (non-perturbatively defined) string theory or M-theory. It is therefore important to check this highly non-trivial proposal. For instance, the gravity dual analysis implies that there should be $\mathcal{O}(N^3)$ degrees of freedom at large N , instead of N^2 as it is naively expected from a 5d gauge theory. So far, there are some indications which support the proposal. These indications are from, for example, the analysis of the M2-M5 bound state [13, 14] and the instanton index counting in flat 5d [15]. But despite these indications, there has not yet been enough evidence for the proposal. The exact partition function for 5d SUSY gauge theories on S^5 will serve as a more direct measure of the degrees of freedom, like the 3d counterpart which was successfully applied to the case of multiple M2-branes [16, 17, 18].

In this paper, we construct the (Euclidean) five-dimensional $\mathcal{N} = 1$ SUSY gauge theory on S^5 with vector and hypermultiplets. This is not a conformal field theory, thus there are 8 SUSY generators and the $SU(2)$ R-symmetry is broken to $U(1)$ by the curvature of S^5 . These include the analogue of the $\mathcal{N} = 2$ SUSY 5d gauge theory on \mathbb{R}^5 for the one on S^5 , i.e. a vector multiplet and an adjoint hypermultiplet. The SUSY transformation and the action are determined from the standard Noether procedure. We also show that, as suggested by [19], the rigid SUSY transformation and the action can be obtained from the off-shell $(4 + 1)$ d supergravity theory [20, 21] by choosing the VEV of the fields in the supergravity multiplet. Indeed, by an appropriate choice of the auxiliary field we show that this also gives the same 5d SUSY gauge theory on S^5 for the vector multiplet.

In order to apply the localization technique, one needs to choose a SUSY generator, which is equivalent to choosing a $SU(2)_R$ doublet spinor. We show that there is essentially only one choice. The bi-linear of the spinors is a vector field without fixed points, which leads to a S^1 fibration over \mathbb{CP}^2 . Then, for the vector multiplet the standard term for localization [3] restricts the path-integral to an integration over a generalization of instantons on \mathbb{CP}^2 and the covariantly constant Coulomb moduli. Unfortunately, we have not succeeded in evaluating the localized path-integral for now, but hope to return to this problem in the near future.

The organization of this paper is as follows: section 2 gives a short review of the 5d spinor calculus and then constructs the 5d SUSY gauge theory on S^5 with vector multiplets and hypermultiplets. Explicit Killing spinors are given at the end of the section. In section 3, localization is applied to the 5d gauge theories on S^5 . We conclude with a short discussion in section 4.

2. 5D SUSY gauge theory on S^5

The 5d SUSY gauge theory on S^5 is constructed in this section.

2.1 5D Spinor Calculus

First, we summarize the properties of the Euclidean 5d spinors in \mathbb{R}^5 . The 5d Gamma matrices are a set of 4×4 hermitian matrices satisfying $\{\Gamma^m, \Gamma^n\} = 2\delta^{mn}$. The standard notation for their antisymmetrized products is

$$\Gamma^{n_1 n_2 \dots n_p} \equiv \frac{1}{p!} (\Gamma^{n_1} \Gamma^{n_2} \dots \Gamma^{n_p} \pm \dots) .$$

We also need a matrix C which relates Γ^m to its transpose as $C\Gamma^m C^{-1} = \pm(\Gamma^m)^T$, called the charge conjugation matrix. Assuming C to be either symmetric or antisymmetric, one easily finds that $C\Gamma^{n_1 \cdots n_p}$ all have definite parity under transposition. Using

$$\Gamma^{n_1 n_2 n_3 n_4 n_5} = \varepsilon^{n_1 n_2 n_3 n_4 n_5} ,$$

one can argue that C and $C\Gamma^m$ are antisymmetric whereas $C\Gamma^{mn}$ are symmetric. They span the 16-dimensional linear space of 4×4 matrices. Accordingly, one also has

$$C\Gamma^m C^{-1} = (\Gamma^m)^T = (\Gamma^m)^* .$$

The above further implies that $(C^* C)\Gamma^m (C^* C)^{-1} = \Gamma^m$ and that $C^* C = -C^\dagger C$ is an Hermitian matrix which commutes with any 4×4 matrix; i.e. it is proportional to the identity. We normalize with

$$C^* C = -1 .$$

In our convention the matrices have the following index structure

$$C_{\alpha\beta} , \quad (\Gamma^m)_{\alpha\beta} , \quad (C\Gamma^{n_1 \cdots n_p})_{\alpha\beta} , \quad -(C^*)^{\alpha\beta} = (C^{-1})^{\alpha\beta} \equiv C^{\alpha\beta} . \quad (2.1)$$

Spinors ψ^α belong to a 4-dimensional representation of the rotation group $Sp(2) \simeq SO(5)$. This representation is pseudoreal, and $(\psi^*)_\alpha$ and $(C\psi)_\alpha \equiv C_{\alpha\beta}\psi^\beta$ transform the same way under rotations. Because it is the pseudoreal representation, the Majorana (or real) condition $\psi^* \equiv C\psi$ does not satisfy the consistency $** = \text{id}$ due to $C^* C = -1$.

The 5d SUSY algebra has $SU(2)$ R-symmetry as an automorphism. For spinors which are doublets under $SU(2)_R$, one can instead impose the $SU(2)$ Majorana condition²,

$$(\psi_I^\alpha)^* = \epsilon^{IJ} C_{\alpha\beta} \psi_J^\beta . \quad (2.2)$$

since they are in real(-positive) representations of $SU(2) \times Spin(5)$. Here, ϵ^{IJ} is the antisymmetric $SU(2)$ invariant tensor defined by $\epsilon^{12} = -\epsilon^{21} = 1$. We also introduce ϵ_{IJ} with $\epsilon_{12} = -\epsilon_{21} = -1$.

For bilinear spinors, we use the following notation

$$\xi\eta \equiv C_{\alpha\beta} \xi^\alpha \eta^\beta , \quad \xi\Gamma^{n_1 \cdots n_p} \eta \equiv (C\Gamma^{n_1 \cdots n_p})_{\alpha\beta} \xi^\alpha \eta^\beta . \quad (2.3)$$

2.2 Vector Multiplets

In this subsection, we concentrate on vector multiplets for an arbitrary gauge group. We first review the 5d SUSY gauge theory on flat \mathbb{R}^5 with off-shell component fields and then study the theory on S^5 .

²Here we are considering Euclidean instead of Lorentzian signature, but we still call this $SU(2)$ Majorana. Note that in the Lorentzian signature there are also $SU(2)$ Majorana spinors.

Flat \mathbb{R}^5 . A vector multiplet contains a 5d vector A_m , a real scalar σ , a triplet of auxiliary scalars D_{IJ} satisfying $(D_{IJ})^\dagger = D^{IJ} \equiv \epsilon^{II'} \epsilon^{JJ'} D_{JJ'}$, and an $SU(2)$ Majorana spinor λ_I^α . For non-abelian gauge symmetry, we assume these fields are Hermitian matrix-valued. On flat \mathbb{R}^5 , their SUSY variation takes the form

$$\begin{aligned}\delta_\xi A_m &= i\epsilon^{IJ} \xi_I \Gamma_m \lambda_J , \\ \delta_\xi \sigma &= i\epsilon^{IJ} \xi_I \lambda_J , \\ \delta_\xi \lambda_I &= -\frac{1}{2} \Gamma^{mn} \xi_I F_{mn} + \Gamma^m \xi_I D_m \sigma + \xi_J D_{KI} \epsilon^{JK} , \\ \delta_\xi D_{IJ} &= -i(\xi_I \Gamma^m D_m \lambda_J + \xi_J \Gamma^m D_m \lambda_I) + [\sigma, \xi_I \lambda_J + \xi_J \lambda_I] ,\end{aligned}\tag{2.4}$$

where we used

$$\begin{aligned}F_{mn} &= \partial_m A_n - \partial_n A_m - i[A_m, A_n] , \\ D_m \sigma &= \partial_m \sigma - i[A_m, \sigma] .\end{aligned}\tag{2.5}$$

These transformation laws are consistent with Hermite conjugation as one can check by using $(\Gamma^m)^* = C\Gamma^m C^{-1}$ along with

$$(\xi_I \lambda_J)^\dagger = -\epsilon^{II'} \epsilon^{JJ'} \xi_{I'} \lambda_{J'} , \quad (\xi_I \Gamma_m \lambda_J)^\dagger = -\epsilon^{II'} \epsilon^{JJ'} \xi_{I'} \Gamma_m \lambda_{J'} ,\tag{2.6}$$

$$(\epsilon^{IJ} \xi_I \lambda_J)^\dagger = -\epsilon^{IJ} \xi_I \lambda_J , \quad (\epsilon^{IJ} \xi_I \Gamma_m \lambda_J)^\dagger = -\epsilon^{IJ} \xi_I \Gamma_m \lambda_J ,\tag{2.7}$$

where we assumed that ξ_I is an $SU(2)$ Majorana fermion. The coefficients of various terms are determined by requiring that the commutator of two SUSY yields the Lie derivative $\mathcal{L}(-iv)$ and gauge transformation \mathcal{G} ,

$$[\delta_\xi, \delta_\eta] = \mathcal{L}(-iv) + \mathcal{G}(\gamma + iv^m A_m) ,\tag{2.8}$$

with

$$v^m = 2\epsilon^{IJ} \xi_I \Gamma^m \eta_J , \quad \gamma = -2i\epsilon^{IJ} \xi_I \eta_J \sigma .\tag{2.9}$$

More explicitly,

$$\begin{aligned}[\delta_\xi, \delta_\eta] A_m &= -iv^m F_{nm} + D_m \gamma , \\ [\delta_\xi, \delta_\eta] \sigma &= -iv^n D_n \sigma , \\ [\delta_\xi, \delta_\eta] \lambda_I &= -iv^n D_n \lambda_I + i[\gamma, \lambda_I] , \\ [\delta_\xi, \delta_\eta] D_{IJ} &= -iv^n D_n D_{IJ} + i[\gamma, D_{IJ}] .\end{aligned}\tag{2.10}$$

In order to derive the above, one needs make use of the Fierz identity which holds for any three spinors $(\xi, \eta, \psi)^3$,

$$\xi^\alpha (\eta \psi) = -\frac{1}{4} \psi^\alpha (\eta \xi) - \frac{1}{4} (\Gamma^m \psi)^\alpha (\eta \Gamma_m \xi) + \frac{1}{8} (\Gamma^{mn} \psi)^\alpha (\eta \Gamma_{mn} \xi) .\tag{2.12}$$

³This is valid if (ξ, η, ψ) are fermions. For bosonic spinors, we have

$$\xi^\alpha (\eta \psi) = \frac{1}{4} \psi^\alpha (\eta \xi) + \frac{1}{4} (\Gamma^m \psi)^\alpha (\eta \Gamma_m \xi) - \frac{1}{8} (\Gamma^{mn} \psi)^\alpha (\eta \Gamma_{mn} \xi) .\tag{2.11}$$

Accordingly, the Yang-Mills term $\frac{1}{2}\text{tr}(F_{mn}F^{mn})$ has the following SUSY completion

$$\mathcal{L}_{\text{SYM}} = \text{tr} \left[\frac{1}{2} F_{mn} F^{mn} - D_m \sigma D^m \sigma - \frac{1}{2} D_{IJ} D^{IJ} + i\epsilon^{IJ} \lambda_I \Gamma^m D_m \lambda_J - \epsilon^{IJ} \lambda_I [\sigma, \lambda_J] \right] . \quad (2.13)$$

Five-Sphere. For SUSY theories on S^5 , the supersymmetry transformation parameter ξ_I is expected to be a Killing spinor satisfying

$$D_m \xi_I \equiv \left(\partial_m + \frac{1}{4} \omega_m^{ab} \Gamma^{ab} \right) \xi_I = \Gamma_m \tilde{\xi}_I \quad (2.14)$$

with a certain $\tilde{\xi}_I$ from the 4d and 3d computations [3] [8]. Here D_m is the local Lorentz covariant derivative and ω_m^{ab} is the spin connection. We also need to distinguish the curved indices (m, n, \dots) and flat indices (a, b, \dots) . Γ^a is constant but $\Gamma_m = e_m^a \Gamma^a$ is coordinate-dependent. We will show that $\tilde{\xi}$ will be given by ξ . In section 2.4, we explicitly construct the Killing spinors on S^5 .

The SUSY variation of fields on S^5 takes the form

$$\begin{aligned} \delta_\xi A_m &= i\epsilon^{IJ} \xi_I \Gamma_m \lambda_J , \\ \delta_\xi \sigma &= i\epsilon^{IJ} \xi_I \lambda_J , \\ \delta_\xi \lambda_I &= -\frac{1}{2} \Gamma^{mn} \xi_I F_{mn} + \Gamma^m \xi_I D_m \sigma + \xi_J D_{KI} \epsilon^{JK} + 2\tilde{\xi}_I \sigma , \\ \delta_\xi D_{IJ} &= -i(\xi_I \Gamma^m D_m \lambda_J + \xi_J \Gamma^m D_m \lambda_I) + [\sigma, \xi_I \lambda_J + \xi_J \lambda_I] + i(\tilde{\xi}_I \lambda_J + \tilde{\xi}_J \lambda_I) . \end{aligned} \quad (2.15)$$

This form is determined from the requirement that the commutator of two SUSY should be a sum of translation (v^m), gauge transformation ($\gamma + iv^m A_m$), dilation (ρ), R-rotation (R_{IJ}) and Lorentz rotation (Θ^{ab}):

$$\begin{aligned} [\delta_\xi, \delta_\eta] A_m &= -iv^n F_{nm} + D_m \gamma , \\ [\delta_\xi, \delta_\eta] \sigma &= -iv^n D_n \sigma + \rho \sigma , \\ [\delta_\xi, \delta_\eta] \lambda_I &= -iv^n D_n \lambda_I + i[\gamma, \lambda_I] + \frac{3}{2} \rho \lambda_I + R_I^J \lambda_J + \frac{1}{4} \Theta^{ab} \Gamma^{ab} \lambda , \\ [\delta_\xi, \delta_\eta] D_{IJ} &= -iv^n D_n D_{IJ} + i[\gamma, D_{IJ}] + 2\rho D_{IJ} + R_I^K D_{KJ} + R_J^K D_{IK} \\ &\quad - 2i\sigma(\eta_I \Gamma^m D_m \tilde{\xi}_J + \eta_J \Gamma^m D_m \tilde{\xi}_I - \xi_I \Gamma^m D_m \tilde{\eta}_J - \xi_J \Gamma^m D_m \tilde{\eta}_I) . \end{aligned} \quad (2.16)$$

Here $R_I^J = \epsilon^{JK} R_{IK}$ and

$$\begin{aligned} v^m &= 2\epsilon^{IJ} \xi_I \Gamma^m \eta_J , \\ \gamma &= -2i\epsilon^{IJ} \xi_I \eta_J \sigma , \\ \rho &= -2i\epsilon^{IJ} (\xi_I \tilde{\eta}_J - \eta_I \tilde{\xi}_J) , \\ R_{IJ} &= -3i(\xi_I \tilde{\eta}_J + \xi_J \tilde{\eta}_I - \eta_I \tilde{\xi}_J - \eta_J \tilde{\xi}_I) , \\ \Theta^{ab} &= -2i\epsilon^{IJ} (\tilde{\xi}_I \Gamma^{ab} \eta_J - \tilde{\eta}_I \Gamma^{ab} \xi_J) . \end{aligned} \quad (2.17)$$

The unwanted last term in the right hand side of $[\delta_\xi, \delta_\eta]D_{IJ}$ vanishes if we require

$$\Gamma^m D_m \tilde{\xi}_I = h \cdot \xi_I \quad \text{or equivalently} \quad \Gamma^m D_m \Gamma^n D_n \xi_I = 5h \cdot \xi_I \quad (2.18)$$

for a certain scalar function h . Note that this also implies

$$\Gamma^{mn} D_m D_n \xi_I = \frac{1}{8} \Gamma^{mn} \Gamma^{ab} R_{mn}{}^{ab} \xi_I = -\frac{1}{4} R \cdot \xi_I = 4h \cdot \xi_I \quad (2.19)$$

For round S^5 with radius ℓ , the scalar curvature is $R = \frac{20}{\ell^2}$ so that $h = -\frac{5}{4\ell^2}$.

The Lagrangian \mathcal{L}_{SYM} for the flat space is not invariant under (2.15) as it is. To see this, we take $\delta\mathcal{L}_{\text{SYM}}$ and extract the term containing the auxiliary field D_{IJ} ,

$$\delta\mathcal{L}_{\text{SYM}} \Big|_{\mathcal{O}(D_{IJ})} = -2i\text{tr}(D^{IJ} \tilde{\xi}_I \lambda_J) \quad (2.20)$$

This can be cancelled by requiring the supersymmetry parameter to satisfy

$$\tilde{\xi}_I = t_I^J \xi_J \quad \text{i.e.} \quad D_m \xi_I = \Gamma_m t_I^J \xi_J \quad (2.21)$$

and by modifying the Lagrangian with the terms

$$\mathcal{L}'_{\text{SYM}} = -it^{IJ} \lambda_I \lambda_J + 2\sigma t^{IJ} D_{IJ} \quad (2.22)$$

Our convention is $t_I^J \equiv \epsilon^{JK} t_{IK}$. The $SU(2)$ Majorana condition on $\xi_I, \tilde{\xi}_I$ leads to

$$(t_{IJ})^* = \epsilon^{II'} \epsilon^{JJ'} t_{I'J'} \quad (2.23)$$

Equivalently, t_I^J as a 2×2 matrix is a linear sum of Pauli's matrices with pure imaginary coefficients. One also finds

$$t_I^J t_J^K = -\frac{1}{4\ell^2} \delta_I^K \quad (2.24)$$

Thus, we can choose, for example,

$$t_I^J = \frac{i}{2\ell} \sigma_3 \quad (2.25)$$

We note that $SU(2)$ R-symmetry is broken by nonzero t_I^J to $U(1)$. We also note that the SUSY algebra does not contain dilatation and $SU(2)$ R-symmetry except for the unbroken $U(1)$. This is seen from (2.17).

Further computation shows

$$\delta(\mathcal{L}_{\text{SYM}} + \mathcal{L}'_{\text{SYM}}) = -20it_I^J t_J^I \text{tr}(\sigma \epsilon^{IJ} \xi_I \lambda_J) = 10t^{IJ} t_{IJ} \delta\text{tr}(\sigma^2) \quad (2.26)$$

so that the invariant Lagrangian is

$$\begin{aligned} \mathcal{L}_{S^5} &= \mathcal{L}_{\text{SYM}} + \mathcal{L}'_{\text{SYM}} - 10t^{IJ} t_{IJ} \text{tr}(\sigma^2) \\ &= \text{tr} \left[\frac{1}{2} F_{mn} F^{mn} - D_m \sigma D^m \sigma - \frac{1}{2} D_{IJ} D^{IJ} + 2\sigma t^{IJ} D_{IJ} - 10t^{IJ} t_{IJ} \sigma^2 \right. \\ &\quad \left. + i\epsilon^{IJ} \lambda_I \Gamma^m D_m \lambda_J - \epsilon^{IJ} \lambda_I [\sigma, \lambda_J] - it^{IJ} \lambda_I \lambda_J \right] \quad (2.27) \end{aligned}$$

From Supergravity. The rigid SUSY theories on curved space can be obtained from the corresponding supergravity theory [19]. The same applies for SUSY gauge theories on S^5 . We start from off-shell 5d supergravity coupled to Yang-Mills theory [20, 21]. We give nonzero VEV to the $SU(2)_R$ triplet auxiliary boson t_{ij} and the metric in the Weyl multiplet which includes the graviton and gravitino. This is because the VEV of the scalar will be needed to obtain S^5 and the only scalar which appears in the SUSY transformation of the gravitino is t_{ij} . This also means that the $SU(2)$ R-symmetry should be broken to $U(1)$ on S^5 . Then, the SUSY transformation of the multiplets vanishes if

$$\delta\Psi_{mI} \sim D_m \xi_I + \Gamma_m t'^J_I \xi_J = 0, \quad (2.28)$$

where Ψ_{mI} is the gravitino. Given that the auxiliary field satisfies the reality condition in (2.23), this is the Killing spinor subsection if one replaces $t'^J_I \rightarrow -t'^J_I$. Accordingly, the SUSY transformation for the vector multiplet is

$$\begin{aligned} \delta_\xi A'_m &= -2i\epsilon^{IJ}\xi_I\Gamma_m\lambda'_J, \\ \delta_\xi\sigma' &= 2i\epsilon^{IJ}\xi_I\lambda'_J, \\ \delta_\xi\lambda'_I &= \frac{1}{4}\Gamma^{mn}\xi_I F'_{mn} + \frac{1}{2}\Gamma^m\xi_I D_m\sigma' - Y_I^J\xi_J, \\ \delta_\xi Y_{IJ} &= i(\xi_I\Gamma^m D_m\lambda'_J + \xi_J\Gamma^m D_m\lambda'_I) + [\sigma', \xi_I\lambda'_J + \xi_J\lambda'_I] + i(t'^K_I \xi_K\lambda'_J + t'^K_J \xi_K\lambda'_I), \end{aligned} \quad (2.29)$$

where we have used the identity

$$\bar{\xi}^I t'^J_K \lambda'^K + \bar{\xi}^J t'^I_K \lambda'^K + 2t'^{IJ}\bar{\xi}^K \lambda_K = (\bar{\xi}^K t'^I_K \lambda'^J + \bar{\xi}^K t'^J_K \lambda'^I). \quad (2.30)$$

The action is

$$\begin{aligned} g'^2 \mathcal{L}'_{S^5} &= \text{tr} \left[\frac{1}{4} F_{mn} F^{mn} - \frac{1}{2} D_m \sigma' D^m \sigma' - Y_{IJ} Y^{IJ} + 4\sigma' t'^{IJ} Y_{IJ} - 8t'^{IJ} t'_{IJ} \sigma'^2 \right. \\ &\quad \left. + 2i\lambda'_I (\epsilon^{IJ}\Gamma^m D_m + t'^{IJ}) \lambda'_J - 2\epsilon^{IJ} [\lambda'_I, \lambda'_J] \sigma' \right], \end{aligned} \quad (2.31)$$

where g' is the gauge coupling constant.⁴

One can show from the result above that under the map

$$\begin{aligned} \sigma &= -\sigma', \\ \lambda_I &= -2\lambda'_I, \\ D_{IJ} + 2t_{IJ}\sigma &= 2Y_{IJ}, \\ t_{IJ} &= -t'_{IJ}, \\ \mathcal{L}_{S^5} &= 2g'^2 \mathcal{L}'_{S^5}, \end{aligned} \quad (2.32)$$

the SUSY transformation in (2.15) and the action in (2.27) are indeed equal to the ones derived from supergravity in (2.29) and (2.31). Using the Chern-Simons term [20], one can construct the SUSY Chern-Simons term on S^5 . We have left out the explicit construction in this paper.

⁴We renormalized all the fields in order to factor out g' . Because [20, 21] used the Lorentz signature, there might be some ambiguities for the Wick rotation, which we fix appropriately here.

For abelian gauge group, FI terms are also SUSY invariant. On flat \mathbb{R}^5 it is given by

$$\mathcal{L}_{\text{FI}} = x^{IJ} D_{IJ}, \quad (2.33)$$

where x^{IJ} is an arbitrary $SU(2)_R$ -triplet constant. On S^5 , one finds that the FI coupling x^{IJ} has to be proportional to t^{IJ} and an improvement term must be added.

$$\mathcal{L}_{\text{FI}} = t^{IJ} D_{IJ} - 6t^{IJ} t_{IJ} \sigma. \quad (2.34)$$

2.3 Hypermultiplets

In this section, we present the SUSY theories with hypermultiplets. The system of r hypermultiplets consists of scalars q_I^A , fermions ψ^A and auxiliary scalars F_I^A . Here, $I = 1, 2$ is the $SU(2)$ R-symmetry index and $A = 1, \dots, 2r$. The fields obey the reality conditions

$$(q_I^A)^* = \Omega_{AB} \epsilon^{IJ} q_J^B, \quad (\psi^{A\alpha})^* = \Omega_{AB} C_{\alpha\beta} \psi^{B\beta}, \quad (F_I^A)^* = \Omega_{AB} \epsilon^{IJ} F_J^B, \quad (2.35)$$

where $\epsilon^{IJ}, C_{\alpha\beta}, \Omega_{AB}$ are antisymmetric invariant tensors of $SU(2) \simeq Sp(1), Spin(5) \simeq Sp(2)$ and the ‘‘flavor symmetry’’ of r free hypermultiplets $Sp(r)$. The coupling to vector multiplets can be introduced via gauging a subgroup of $Sp(r)$.

Flat \mathbb{R}^5 . It is said that one cannot realize off-shell supersymmetry on hypermultiplets with a finite number of auxiliary fields. Let us review this by first studying the free theory on \mathbb{R}^5 .

It can be easily shown that the Lagrangian

$$\mathcal{L} = \epsilon^{IJ} \Omega_{AB} \partial_m q_I^A \partial^m q_J^B - 2i \Omega_{AB} \psi^A \Gamma^m \partial_m \psi^B \quad (2.36)$$

is invariant under the on-shell supersymmetry transformation

$$\delta q_I^A = -2i \xi_I \psi^A, \quad \delta \psi^A = \epsilon^{IJ} \Gamma^m \xi_I \partial_m q_J^A. \quad (2.37)$$

The commutator of two supersymmetries acts on the fields as

$$\begin{aligned} [\delta_\xi, \delta_\eta] q_I^A &= -2i \epsilon^{JK} \xi_J \Gamma^m \eta_K \cdot \partial_m q_I^A, \\ [\delta_\xi, \delta_\eta] \psi^A &= -2i \epsilon^{IJ} \Gamma^m \eta_I \cdot \xi_J \partial_m \psi^A - (\xi \leftrightarrow \eta) \\ &= -i \partial_m \psi^A \cdot \epsilon^{IJ} \xi_I \Gamma^m \eta_J - i \Gamma^m \partial_m \psi^A \cdot \epsilon^{IJ} \xi_I \eta_J + i \Gamma^{\ell m} \partial_m \psi^A \cdot \epsilon^{IJ} \xi_I \Gamma_\ell \eta_J \\ &= -2i \partial_m \psi^A \cdot \epsilon^{IJ} \xi_I \Gamma^m \eta_J + \Delta \psi^A. \end{aligned} \quad (2.38)$$

Here,

$$\Delta \psi^A \equiv -2i \Gamma^m \partial_m \psi^A \cdot \epsilon^{IJ} \xi_I \eta_J + 2i \eta_I \cdot \epsilon^{IJ} \xi_J \Gamma^m \partial_m \psi^A - 2i \xi_I \cdot \epsilon^{IJ} \eta_J \Gamma^m \partial_m \psi^A, \quad (2.39)$$

and in the last equality we used

$$\begin{aligned} &-2i \epsilon^{IJ} \eta_I \cdot \xi_J \Gamma^m \partial_m \psi^A - (\xi \leftrightarrow \eta) \\ &= -i \partial_m \psi^A \cdot \epsilon^{IJ} \xi_I \Gamma^m \eta_J - i \Gamma^m \partial_m \psi^A \cdot \epsilon^{IJ} \xi_I \eta_J - i \Gamma^{\ell m} \partial_m \psi^A \cdot \epsilon^{IJ} \xi_I \Gamma_\ell \eta_J. \end{aligned} \quad (2.40)$$

The commutator of two supersymmetries therefore does not precisely close under a translation by $v^m \equiv 2\epsilon^{JK}\xi_J\Gamma^m\eta_K$. The failure terms in $\Delta\psi^A$ are all proportional to the equation of motion $\Gamma^m\partial_m\psi^A$.

One can try to modify the supersymmetry transformation law by introducing the auxiliary field F_I^A . From dimensional counting and the symmetries, the only sensible generalization is

$$\delta q_I^A \stackrel{?}{=} -2i\xi_I\psi^A, \quad \delta\psi^A \stackrel{?}{=} \epsilon^{IJ}\Gamma^m\xi_I\partial_m q_J^A + \alpha\epsilon^{IJ}\xi_I F_J^A, \quad (2.41)$$

with an unknown parameter α . But whatever the value of α is, it leads to a failure of the closure of supersymmetry commutators on q_I^A ,

$$[\delta_\xi, \delta_\eta]q_I^A \stackrel{?}{=} -2i\epsilon^{JK}\xi_J\Gamma^m\eta_K \cdot \partial_m q_I^A - 2i\alpha\epsilon^{JK}\xi_J\eta_K \cdot F_I^A. \quad (2.42)$$

Thus we will not try to find the transformation law which satisfies that $[\delta_\xi, \delta_\eta]$ is a translation for any pair (ξ, η) . We rather look for the transformation law δ which satisfies that δ^2 is a translation for any bosonic ξ .⁵ This property is sufficient for localization. We propose

$$\begin{aligned} \delta q_I^A &= -2i\xi_I\psi^A, \\ \delta\psi^A &= \epsilon^{IJ}\Gamma^m\xi_I\partial_m q_J^A + \epsilon^{I'J'}\check{\xi}_{I'}F_{J'}^A, \\ \delta F_{I'}^A &= 2i\check{\xi}_{I'}\Gamma^m\partial_m\psi^A. \end{aligned} \quad (2.43)$$

and an invariant Lagrangian

$$\mathcal{L} = \epsilon^{IJ}\Omega_{AB}\partial_m q_I^A\partial_m q_J^B - 2i\Omega_{AB}\psi^A\Gamma^m\partial_m\psi^B - \epsilon^{I'J'}\Omega_{AB}F_{I'}^A F_{J'}^B. \quad (2.44)$$

Here, $\check{\xi}_{I'}$ is a constant spinor which satisfies

$$\epsilon^{IJ}\xi_I\xi_J = \epsilon^{I'J'}\check{\xi}_{I'}\check{\xi}_{J'}, \quad \xi_I\check{\xi}_{J'} = 0, \quad \epsilon^{IJ}\xi_I\Gamma^m\xi_J + \epsilon^{I'J'}\check{\xi}_{I'}\Gamma^m\check{\xi}_{J'} = 0. \quad (2.45)$$

It looks nontrivial that a spinor $\check{\xi}_{I'}$ exists for any choice of ξ_I . Therefore, let us prove its existence here. First, given a pair (ξ_1, ξ_2) of 4-component spinors with the skew-symmetric inner product $\xi_1\xi_2 \equiv C_{\alpha\beta}\xi_1^\alpha\xi_2^\beta = 1$, it is elementary that one can find two more spinors $\check{\xi}_1, \check{\xi}_2$ satisfying

$$\check{\xi}_{I'}\xi_J = 0, \quad \check{\xi}_1\check{\xi}_2 = 1.$$

Then the tracelessness of Γ^m in the basis $\xi_1, \xi_2, \check{\xi}_1, \check{\xi}_2$ gives

$$\begin{aligned} 0 &= \text{Tr}\Gamma^m = \xi_1\Gamma^m\xi_2 - \xi_2\Gamma^m\xi_1 + \check{\xi}_1\Gamma^m\check{\xi}_2 - \check{\xi}_2\Gamma^m\check{\xi}_1 \\ &= \epsilon^{IJ}\xi_I\Gamma^m\xi_J + \epsilon^{I'J'}\check{\xi}_{I'}\Gamma^m\check{\xi}_{J'}. \end{aligned} \quad (2.46)$$

We note that the action in (2.44) is invariant under the SUSY transformation with any Killing spinor ξ_I and corresponding $\check{\xi}_I$. Thus it gives a 5d $\mathcal{N} = 1$ SUSY theory with 8 SUSY generators although the commutators between them include terms other than the (usual) symmetries of the theories. Furthermore, there is an additional $SU(2)'$ symmetry, which acts on I', J' indices.

⁵Below, we will denote δ as a fermionic transformation generated by a Grassmann-even Killing spinor ξ . This notation will be used for the localization.

To introduce the coupling to gauge fields and other fields in the vector multiplet, we need first to introduce the covariant derivative

$$D_m \psi^A \equiv \partial_m \psi^A - i(A_m)^A_B \psi^B, \text{ etc.} \quad (2.47)$$

Requiring Ω_{AB} to be gauge-invariant, one finds $(A_m)_{AB} \equiv \Omega_{AC}(A_m)^C_B$ to be symmetric in the indices A, B . In the following, we introduce the notation $\bar{\psi}_B \equiv \psi^A \Omega_{AB}$ and suppress the indices A, B, \dots , such that

$$\begin{aligned} \epsilon^{IJ} \Omega_{AB} D_m q_I^A D^m q_J^B &\equiv \epsilon^{IJ} D_m \bar{q}_I D^m q_J, \\ \Omega_{AB} \psi^A \Gamma^m (A_m)^B_C \psi^C &\equiv \bar{\psi} \Gamma^m A_m \psi, \text{ etc.} \end{aligned} \quad (2.48)$$

The invariant Lagrangian is

$$\begin{aligned} \mathcal{L} = & \epsilon^{IJ} (D_m \bar{q}_I D^m q_J - \bar{q}_I \sigma^2 q_J) - 2(i\bar{\psi} \Gamma^m D_m \psi + \bar{\psi} \sigma \psi) \\ & - i\bar{q}_I D^{IJ} q_J - 4\epsilon^{IJ} \bar{\psi} \lambda_I q_J - \epsilon^{I'J'} \bar{F}_{I'} F_{J'} . \end{aligned} \quad (2.49)$$

The corresponding SUSY transformation is

$$\begin{aligned} \delta q_I &= -2i\xi_I \psi, \\ \delta \psi &= \epsilon^{IJ} \Gamma^m \xi_I D_m q_J + i\epsilon^{IJ} \xi_I \sigma q_J + \epsilon^{I'J'} \check{\xi}_{I'} F_{J'}, \\ \delta F_{I'} &= 2\check{\xi}_{I'} (i\Gamma^m D_m \psi + \sigma \psi + \epsilon^{KL} \lambda_K q_L). \end{aligned} \quad (2.50)$$

Five-Sphere. Let us first consider the system of free hypermultiplets on S^5 . We find that the Lagrangian

$$\mathcal{L} = \epsilon^{IJ} \Omega_{AB} D_m q_I^A D^m q_J^B - 2i\Omega_{AB} \psi^A \Gamma^m D_m \psi^B + \frac{15}{2} \epsilon^{IJ} \Omega_{AB} t^{KL} t_{KL} q_I^A q_J^B \quad (2.51)$$

is invariant under the on-shell transformation law

$$\delta q_I^A = -2i\xi_I \psi^A, \quad \delta \psi^A = \epsilon^{IJ} \Gamma^m \xi_I D_m q_J^A - 3t^{IJ} \xi_I q_J^A. \quad (2.52)$$

Then the unique off-shell extension is given by the Lagrangian

$$\begin{aligned} \mathcal{L} = & \epsilon^{IJ} \Omega_{AB} D_m q_I^A D^m q_J^B - 2i\Omega_{AB} \psi^A \Gamma^m D_m \psi^B \\ & + \frac{15}{2} \epsilon^{IJ} \Omega_{AB} t^{KL} t_{KL} q_I^A q_J^B - \epsilon^{I'J'} \Omega_{AB} F_{I'}^A F_{J'}^B, \end{aligned} \quad (2.53)$$

and the transformation law

$$\begin{aligned} \delta q_I^A &= -2i\xi_I \psi^A, \\ \delta \psi^A &= \epsilon^{IJ} \Gamma^m \xi_I D_m q_J^A - 3t^{IJ} \xi_I q_J^A + \epsilon^{I'J'} \check{\xi}_{I'} F_{J'}^A, \\ \delta F_{I'}^A &= 2i\check{\xi}_{I'} \Gamma^m D_m \psi^A. \end{aligned} \quad (2.54)$$

For systems coupled to gauge fields, we find that the SUSY invariant Lagrangian is

$$\begin{aligned}\mathcal{L}_{hyper} = & \epsilon^{IJ}(D_m \bar{q}_I D^m q_J - \bar{q}_I \sigma^2 q_J) - 2(i\psi \Gamma^m D_m \psi + \bar{\psi} \sigma \psi) \\ & - i\bar{q}_I D^{IJ} q_J - 4\epsilon^{IJ} \bar{\psi} \lambda_{IJ} q_J + \frac{15}{2} t^{KL} t_{KL} \epsilon^{IJ} \bar{q}_I q_J - \epsilon^{I'J'} \bar{F}_{I'} F_{J'} ,\end{aligned}\quad (2.55)$$

with the associated transformation law being

$$\begin{aligned}\delta q_I &= -2i\xi_I \psi, \\ \delta \psi &= \epsilon^{IJ} \Gamma^m \xi_I D_m q_J + i\epsilon^{IJ} \xi_I \sigma q_J - 3t^{IJ} \xi_I q_J + \epsilon^{I'J'} \tilde{\xi}_{I'} F_{J'}, \\ \delta F_{I'} &= 2\tilde{\xi}_{I'} (i\Gamma^m D_m \psi + \sigma \psi + \epsilon^{KL} \lambda_{KL} q_L).\end{aligned}\quad (2.56)$$

The square of δ is

$$\begin{aligned}\delta^2 q_I &= -iv^m D_m q_I + i\gamma q_I + R_I^J q_J \\ \delta^2 \psi &= -iv^m D_m \psi + i\gamma \psi + \frac{1}{4} \Theta^{ab} \Gamma^{ab} \psi \\ \delta^2 F_{I'} &= -iv^m D_m F_{I'} + i\gamma F_{I'} + R'_{I'}^{J'} F_{J'} ,\end{aligned}\quad (2.57)$$

where

$$\begin{aligned}v^m &= \epsilon^{IJ} \xi_I \Gamma^m \xi_J , \\ \gamma &= -i\epsilon^{IJ} \xi_I \xi_J \sigma , \\ R_{IJ} &= 3i(\epsilon^{KL} \xi_K \xi_L) t_{IJ} , \\ \Theta^{ab} &= -2i\epsilon^{IJ} \tilde{\xi}_I \Gamma^{ab} \xi_J , \\ R'_{I'J'} &= -2i\tilde{\xi}_{I'} \Gamma^m D_m \tilde{\xi}_{J'} .\end{aligned}\quad (2.58)$$

Accordingly, δ^2 is a sum of translation (v^m), gauge transformation ($\gamma + iv^m A_m$), R-rotation (R_{IJ}), Lorentz rotation (Θ^{ab}), and $SU(2)'$ rotation ($R'_{I'J'}$). This is consistent with the δ^2 for the vector multiplets. We can also see that the $R'_{I'J'}$ is indeed in the $SU(2)'$ from the equation $\epsilon^{I'J'} R'_{I'J'} = 0$ which follows from the definition of $\tilde{\xi}_{I'}$ and the Killing spinor equation.

We can now consider the mass term for the hypermultiplets. As it is well-known for 4d $\mathcal{N} = 2$ gauge theories, we can take a decoupling limit of some vector multiplets to obtain the flavor symmetry and mass terms from the VEV of the scalar in the vector multiplet. In our case, we require a constant $m \equiv \langle \sigma \rangle$, $\langle A_m \rangle = 0$, $\langle \lambda \rangle = 0$ and $\langle D_{IJ} \rangle = -2t_{IJ} \langle \sigma \rangle$ for the unbroken SUSY and the bosonic symmetry. Accordingly, the mass term is given from (2.55) as

$$\mathcal{L}_{mass} = -\epsilon^{IJ} \bar{q}_I m^2 q_J - 2\bar{\psi} m \psi + 2it^{IJ} \bar{q}_I m q_J = \bar{q}_I (-\epsilon^{IJ} m^2 + 2it^{IJ} m) q_J - 2\bar{\psi} m \psi. \quad (2.59)$$

We note that m is an abbreviation for m_A^B which should commute with the remaining gauge symmetry. We see that the SUSY transformation law in (2.56) now depends on the mass parameter m even though we are considering the off-shell fields and δ^2 includes the flavor symmetry generator linear in m .

2.4 Killing Spinors on S^5

By now, we have assumed the existence of Killing spinors on S^5 , ξ_I . In this subsection, we construct them explicitly.

Metric. Flat \mathbb{R}^5 and round S^5 (with radius ℓ) have the metrics

$$\begin{aligned} ds_{\mathbb{R}^5}^2 &= \sum_{n=1}^5 dx^n dx^n = dr^2 + r^2 ds_{S^4}^2 , \\ ds_{S^5}^2 &= \ell^2 (d\theta^2 + \sin^2 \theta ds_{S^4}^2) , \end{aligned} \quad (2.60)$$

where $r^2 = \sum_{n=1}^5 (x^n)^2$. One can embed a round S^5 in flat \mathbb{R}^6 , and think of flat \mathbb{R}^5 which contacts the S^5 at its south pole. Then stereographic projection maps every point on the S^5 onto \mathbb{R}^5 by a line passing through the north pole. It gives the relation $r = 2\ell \tan \frac{\theta}{2}$ and

$$\ell^2 d\theta^2 = \frac{dr^2}{(1 + \frac{r^2}{4\ell^2})^2} , \quad \ell^2 \sin^2 \theta = \frac{r^2}{(1 + \frac{r^2}{4\ell^2})^2} . \quad (2.61)$$

Therefore,

$$ds_{S^5}^2 = \frac{dr^2 + r^2 ds_{S^4}^2}{(1 + \frac{r^2}{4\ell^2})^2} = \frac{\sum dx_n^2}{(1 + \frac{r^2}{4\ell^2})^2} = \sum_{a=1}^5 e^a e^a , \quad (2.62)$$

where $e^a = f \delta_n^a dx^n$ and $f = (1 + \frac{r^2}{4\ell^2})^{-1}$. The spin connection $\omega^{ab} \equiv \sum_c \omega^{ab,c} e^c$ is determined from the torsion-free condition

$$0 = de^a + \omega^{ab} e^b = f^{-2} \partial_n f \delta_b^n \cdot e^b e^a - \omega^{ab,c} e^b e^c . \quad (2.63)$$

The corresponding solution is

$$\omega^{ab,c} = f^{-2} \partial_n f (\delta^{ac} \delta^{nb} - \delta^{bc} \delta^{na}) . \quad (2.64)$$

Killing spinor equation. We first solve the Killing spinor equation without the $SU(2)$ R-index:

$$D_m \Psi = \frac{1}{2\ell} \Gamma_m \tilde{\Psi} , \quad (2.65)$$

which becomes

$$D_m \Psi \equiv \partial_m \Psi + \frac{1}{2} \Gamma^{ab} \delta_m^a \delta^{nb} \partial_n \ln f \Psi = \frac{1}{2\ell} f \delta_m^a \Gamma^a \tilde{\Psi} , \quad (2.66)$$

where the coefficient $1/2\ell$ in the right hand side is put for later convenience, and the Gamma matrices are all coordinate independent. The above equation can be rewritten as

$$\partial_m (f^{-\frac{1}{2}} \Psi) = \frac{1}{2\ell} f^{\frac{1}{2}} \delta_m^a \Gamma^a (\tilde{\Psi} + \ell \delta_a^n \Gamma^a \partial_n f^{-1} \Psi) . \quad (2.67)$$

The simplest solution is

$$\Psi = f^{\frac{1}{2}} \Psi_0, \quad \tilde{\Psi} = -\frac{\Gamma^a x^a}{2\ell} f^{\frac{1}{2}} \Psi_0 . \quad (2.68)$$

One can furthermore find

$$D_m \tilde{\Psi} = -\frac{1}{2\ell} f \delta_m^a \Gamma^a \Psi . \quad (2.69)$$

Next, we find the $SU(2)$ Majorana spinor field ξ_I satisfying

$$D_m \xi_I = t_I^J \Gamma_m \xi_J . \quad (2.70)$$

Setting $t_1^1 = -t_2^2 = \frac{i}{2\ell}$, $t_1^2 = t_2^1 = 0$ one obtains

$$\begin{aligned} \xi_1 &= \left(1 + \frac{i\Gamma^a x^a}{2\ell}\right) f^{\frac{1}{2}} \Psi_1 , \\ \xi_2 &= \left(1 - \frac{i\Gamma^a x^a}{2\ell}\right) f^{\frac{1}{2}} \Psi_2 , \end{aligned} \quad (2.71)$$

where Ψ_1, Ψ_2 are constant spinors related to each other by $\Psi_1^* = C\Psi_2$, $\Psi_2^* = -C\Psi_1$.

Bilinears of Killing spinors. The scalar bilinear of the Killing spinors takes the value

$$\epsilon^{IJ} \xi_I \xi_J = 2\xi_1^T C \xi_2 = 2f \Psi_1^T \left(1 + \frac{i(\Gamma^a)^T x^a}{2\ell}\right) C \left(1 - \frac{i\Gamma^a x^a}{2\ell}\right) \Psi_2 = 2\Psi_2^\dagger \Psi_2 , \quad (2.72)$$

Let us normalize it to unity, $2\Psi_2^\dagger \Psi_2 = 1$. The vector bilinear takes the form

$$\begin{aligned} v &\equiv \epsilon^{IJ} \xi_I \Gamma^n \xi_J \frac{\partial}{\partial x^n} = 2f \Psi_1^T \left(1 + \frac{i(\Gamma^b)^T x^b}{2\ell}\right) C \Gamma^n \left(1 - \frac{i\Gamma^c x^c}{2\ell}\right) \Psi_2 \frac{\partial}{\partial x^n} \\ &= 2\Psi_2^\dagger \left(1 + \frac{i\Gamma^b x^b}{2\ell}\right) \Gamma^a \left(1 - \frac{i\Gamma^c x^c}{2\ell}\right) \Psi_2 \frac{\partial}{\partial x^a} . \end{aligned} \quad (2.73)$$

Assuming Ψ_2 to be an eigenspinor for $\Gamma^{12} = \Gamma^{34} = i$ and $\Gamma^5 = -1$, the vector bilinear simplifies to the following form

$$\begin{aligned} v &= -\frac{1}{\ell} \Psi_2^\dagger \Gamma^{12} \Psi_2 (x^1 \partial_2 - x^2 \partial_1) - \frac{1}{\ell} \Psi_2^\dagger \Gamma^{34} \Psi_2 (x^3 \partial_4 - x^4 \partial_3) \\ &\quad - \left\{ \left(1 - \frac{x^2}{4\ell^2}\right) \partial_5 + \frac{x^5 x^a}{2\ell^2} \partial_a \right\} . \end{aligned} \quad (2.74)$$

We can show, when we regard the S^5 as embedded into flat \mathbb{R}^6 , that the above vector bilinear v is a sum of rotations about the 12, 34, 56-planes with an equal angular velocity. In order to show this, we introduce the Cartesian coordinates Y_1, \dots, Y_6 on \mathbb{R}^6 to express the round S^5 as

$$Y_1^2 + \dots + Y_6^2 = \ell^2 , \quad (2.75)$$

with

$$Y_6 = \ell \cos \theta , \quad Y_a = \ell \sin \theta \hat{r}_a , \quad (2.76)$$

where \hat{r}^a is a unit 5-vector. Combining the above with $x_a = 2\ell \tan \frac{\theta}{2} \hat{r}_a$, one finds the relation between the coordinates (x_1, \dots, x_5) and (Y_1, \dots, Y_5) ,

$$Y_a = \frac{x_a}{1 + \frac{x^2}{4\ell^2}} , \quad \frac{\partial}{\partial x_a} = \frac{1}{1 + \frac{x^2}{4\ell^2}} \frac{\partial}{\partial Y_a} - \frac{x_a x_b}{2\ell^2 (1 + \frac{x^2}{4\ell^2})^2} \frac{\partial}{\partial Y_b} . \quad (2.77)$$

Inserting (2.77) into the expression for v in (2.74), one obtains

$$v = \frac{1}{\ell} \left(Y_2 \frac{\partial}{\partial Y_1} - Y_1 \frac{\partial}{\partial Y_2} + Y_4 \frac{\partial}{\partial Y_3} - Y_3 \frac{\partial}{\partial Y_4} + Y_6 \frac{\partial}{\partial Y_5} \right) \quad (2.78)$$

as a vector field on S^5 with the coordinate system (Y_1, \dots, Y_5) . This is a restriction onto the S^5 of a vector field \hat{v} on \mathbb{R}^6 ,

$$\hat{v} = \frac{1}{\ell} \left(Y_2 \frac{\partial}{\partial Y_1} - Y_1 \frac{\partial}{\partial Y_2} + Y_4 \frac{\partial}{\partial Y_3} - Y_3 \frac{\partial}{\partial Y_4} + Y_6 \frac{\partial}{\partial Y_5} - Y_5 \frac{\partial}{\partial Y_6} \right), \quad (2.79)$$

which is the vector field generating the simultaneous rotations about the 12, 34, 56-planes by the same angular velocity. Note that, if we dimensionally reduce the S^5 along v , we obtain \mathbb{CP}^2 .

Circle fibration over \mathbb{CP}^2 . In order to write the metric on S^5 such that the circle fibration structure is manifest, we introduce the (inhomogeneous) complex coordinates z^1, z^2 and an angular coordinate ϑ to express Y_1, \dots, Y_6 as

$$Y_1 + iY_2 = R e^{i\vartheta} z^1, \quad Y_3 + iY_4 = R e^{i\vartheta} z^2, \quad Y_5 + iY_6 = R e^{i\vartheta},$$

$$R \equiv \frac{\ell}{\sqrt{1 + |z^1|^2 + |z^2|^2}}. \quad (2.80)$$

Following the above, the vector field v becomes simply $v = -\frac{1}{\ell} \partial_\vartheta$, and the metric on S^5 reads

$$ds^2 = \sum_{i=1}^6 dY_i dY_i$$

$$= \ell^2 \left[\left(d\vartheta + \frac{i(z^i d\bar{z}^i - \bar{z}^i dz^i)}{2(1 + |z^1|^2 + |z^2|^2)} \right)^2 + \frac{dz^i d\bar{z}^i}{1 + |z^1|^2 + |z^2|^2} - \frac{z^i d\bar{z}^i \cdot dz^j \bar{z}^j}{(1 + |z^1|^2 + |z^2|^2)^2} \right]. \quad (2.81)$$

With the notation

$$ds^2 = \ell^2 [(d\vartheta + V)^2 + 2g_{i\bar{j}} dz^i d\bar{z}^j], \quad V = V_i dz^i + V_{\bar{i}} d\bar{z}^{\bar{i}},$$

$$g_{i\bar{j}} = \frac{1}{2} \partial_i \bar{\partial}_{\bar{j}} \ln(1 + |z^1|^2 + |z^2|^2), \quad (2.82)$$

we have

$$dV = 2ig_{i\bar{j}} dz^i \wedge d\bar{z}^{\bar{j}}. \quad (2.83)$$

If we use the above metric on S^5 , a contravariant vector X has components $X^1, X^2, X^{\bar{1}}, X^{\bar{2}}$ and X^ϑ . The inner product of contravariant vectors are

$$G_{mn} X^m Y^n = \ell^2 \left[(X^\vartheta + V_i X^i + V_{\bar{i}} X^{\bar{i}})(Y^\vartheta + V_i Y^i + V_{\bar{i}} Y^{\bar{i}}) + g_{i\bar{j}}(X^i Y^{\bar{j}} + X^{\bar{j}} Y^i) \right], \quad (2.84)$$

with the component of the metric G_{mn} being

$$G_{\vartheta\vartheta} = \ell^2, \quad G_{\vartheta i} = \ell^2 V_i, \quad G_{\vartheta \bar{i}} = \ell^2 V_{\bar{i}},$$

$$G_{i\bar{j}} = \ell^2 (g_{i\bar{j}} + V_i V_{\bar{j}}), \quad G_{ij} = \ell^2 V_i V_j, \quad G_{\bar{i}\bar{j}} = \ell^2 V_{\bar{i}} V_{\bar{j}}. \quad (2.85)$$

The inverse metric has components

$$\begin{aligned} G^{\vartheta\vartheta} &= \ell^{-2}(1 + 2g^{i\bar{j}}V_iV_{\bar{j}}) , & G^{\vartheta i} &= -\ell^{-2}g^{i\bar{j}}V_{\bar{j}} , & G^{\vartheta\bar{i}} &= -\ell^{-2}g^{j\bar{i}}V_j , \\ G^{i\bar{j}} &= \ell^{-2}g^{i\bar{j}} , & G^{ij} &= G^{\bar{i}\bar{j}} = 0 , \end{aligned} \quad (2.86)$$

where $g^{i\bar{j}}$ is the inverse metric on the base \mathbb{CP}^2 , namely $g_{i\bar{j}}g^{k\bar{j}} = \delta_i^k$.

3. Localization

In this section, we apply localization to the 5d gauge theories.

Vector Multiplets. Let us first concentrate on the vector multiplets and choose a Killing spinor ξ_I . Denoting the corresponding SUSY transformation as δ , one notes that δ^2 is a combination of the transformation generated by v^m and an $U(1)$ R- and Lorentz transformation. Assuming that the transformation δ is the quantum mechanical symmetry, we obtain

$$\frac{d}{dt}\langle \mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_n e^{-t\delta I} \rangle = 0 , \quad (3.1)$$

where \mathcal{O}_i and $I = \int_{S^5} V$ are assumed to satisfy $\delta\mathcal{O}_i = 0$ and

$$\delta^2 I = 0 . \quad (3.2)$$

Here, we also assume that δI is (real) positive definite in the path-integral. Accordingly, by taking $t \rightarrow \infty$, the path-integral is localized on the constraint $\delta I = 0$ with the one loop determinant for the regulator action being $-\delta I$.

To explain our choice of the regulator Lagrangian, we recall

$$\delta\lambda_I = -\frac{1}{2}\Gamma^{mn}\xi_I F_{mn} + \Gamma^m\xi_I D_m\sigma + \xi_J(D_{KI} + \sigma t_{KI})\epsilon^{JK} . \quad (3.3)$$

In this section, we take ξ_I as Grassmann-even such that δ is the fermionic transformation. Of course, this is the symmetry of the action because it is linear in ξ_I . Note that the commutator $[\delta_\xi, \delta_\xi]$ becomes $2\delta^2$. For δ^2 , the right-hand side of the commutators in (2.16) are unchanged, but the parameters become

$$\begin{aligned} v^m &= \epsilon^{IJ}\xi_I\Gamma^m\xi_J , \\ \gamma &= -i\epsilon^{IJ}\xi_I\xi_J\sigma , \\ \rho &= 0 , \\ R_{IJ} &= -3i(\xi_I\tilde{\xi}_J + \xi_J\tilde{\xi}_I) = 3i(\epsilon^{KL}\xi_K\xi_L)t_{IJ} , \\ \Theta^{ab} &= -2i\epsilon^{IJ}\tilde{\xi}_I\Gamma^{ab}\xi_J . \end{aligned} \quad (3.4)$$

To make the SYM Lagrangian positive definite, we notice that the path integration contours for σ and D have to be rotated by 90 degrees, which implies that they are regarded as purely imaginary.⁶ Accordingly, the complex conjugate of the above formula is

$$(\delta\lambda_I)^* = -\frac{1}{2}\epsilon^{II'}C\Gamma^{mn}\xi_{I'}F_{mn} - \epsilon^{II'}C\Gamma^m\xi_{I'}D_m\sigma - \epsilon^{II'}C\xi_J(D_{KI'} + \sigma t_{KI'})\epsilon^{JK} . \quad (3.5)$$

Its transpose is

$$(\delta\lambda_I)^\dagger = +\frac{1}{2}\epsilon^{II'}F_{mn}\xi_{I'}C\Gamma^{mn} - \epsilon^{II'}D_m\sigma\xi_{I'}C\Gamma^m - \epsilon^{II'}(D_{KI'} + \sigma t_{KI'})\xi_JC\epsilon^{JK} . \quad (3.6)$$

We then take the regulator Lagrangian of the form δV , with

$$\begin{aligned} V &= \text{tr}[(\delta\lambda)^\dagger\lambda] \\ &= \text{tr}\left[\frac{1}{2}\epsilon^{IJ}\xi_I\Gamma^{mn}\lambda_JF_{mn} - \epsilon^{IJ}\xi_I\Gamma^m\lambda_JD_m\sigma - \epsilon^{IJ}\xi_K\lambda_J(D_{LI} + 2\sigma t_{LI})\epsilon^{KL}\right] \end{aligned} \quad (3.7)$$

and ξ_I being Grassmann-even.⁷ One should note that

$$\delta^2 \int_{S^5} V = 0 , \quad (3.8)$$

which can be shown as follows. The δ^2 is the bosonic symmetry transformation of (3.4), however, ξ_I , which is not a field, does not transform under the transformation. Since all indices are properly contracted in V , (3.8) is correct if ξ_I would transform under the symmetry as its indices indicate. This is possible if ξ_I is invariant under this. Indeed, the Jacobi identity $[\delta^2, \delta] = 0$ implies that ξ_I is invariant. This is because by decomposing δ to a spinor and to $SU(2)$ components as $\delta = \xi_{\alpha I}\delta^{\alpha I}$, we see that $[\delta^2, \delta] = \delta_{\xi'}$ where ξ' is the transformation of ξ by the bosonic symmetry. We can also show the invariance explicitly by using the identity $t^{KJ}w_{KJ}^{ab}\Gamma_{ab}\xi_I = -4t_I{}^J\xi_J$ followed from the Fierz identity.

δV consists of a collection of purely bosonic terms and terms bilinear in the fermion. The purely bosonic terms read

$$\begin{aligned} \delta V|_{\text{bos}} &= \text{tr}\left[\frac{1}{2}F^{mn}F_{mn} - D_m\sigma D^m\sigma - \frac{1}{2}(D_{IJ} + 2\sigma t_{IJ})(D^{IJ} + 2\sigma t^{IJ}) \right. \\ &\quad \left. - \frac{1}{4}v_p\epsilon^{klmnp}F_{kl}F_{mn}\right] . \end{aligned} \quad (3.9)$$

Using $v_mv^m = 1$ (which is derived below), one can complete the square such that

$$\begin{aligned} \delta V|_{\text{bos}} &= \text{tr}\left[\frac{1}{4}(F_{mn} - \frac{1}{2}\epsilon_{mnpqr}v^pF^{qr})(F^{mn} - \frac{1}{2}\epsilon^{mnstu}v_sF_{tu}) + \frac{1}{2}(v^pF_{pm})(v_qF^{qm}) \right. \\ &\quad \left. - D_m\sigma D^m\sigma - \frac{1}{2}(D_{IJ} + 2\sigma t_{IJ})(D^{IJ} + 2\sigma t^{IJ})\right] . \end{aligned} \quad (3.10)$$

⁶Here, the SUSY action and the SUSY transformation are written in terms of $\lambda_I, \sigma, D_{IJ}$, which are holomorphic. Accordingly, the action is SUSY invariant for any choice of the contour. This is clear because we have not used $\lambda_I^\dagger, \sigma^\dagger, D_{IJ}^\dagger$, in the Lagrangian and in the SUSY transformation. The choice here corresponds to, for example, $\sigma^\dagger = -\sigma$, which is not the relation $\sigma^\dagger = \sigma$ originally assumed.

⁷We can think of the right-hand side as the definition of I and forget about the definition of $\delta(\lambda_I)^\dagger$.

This is indeed positive definite (by definition) for our choice of the contour. The saddle point condition $\delta V|_{bos} = 0$ is therefore

$$F_{mn} = \frac{1}{2}\epsilon_{mnpqr}v^p F^{qr} \ , \quad v^m F_{mn} = 0 \ , \quad D_m \sigma = 0 \ , \quad D_{IJ} + 2\sigma t_{IJ} = 0 \ , \quad (3.11)$$

where the first equation implies the second equation. This kind of instanton equations was studied in [22].

Below, we derive the equality $v_m v^m = 1$. This follows from a stronger equality

$$\Gamma_m \xi_I \cdot v^m = \xi_I \ . \quad (3.12)$$

To show this, we look into the following Fierz identities⁸

$$\begin{aligned} \xi_I &= \xi_I \epsilon^{JK} (\xi_J \xi_K) \\ &= \frac{1}{4} \xi_K \epsilon^{JK} (\xi_J \xi_I) + \frac{1}{4} \Gamma_\ell \xi_K \epsilon^{JK} (\xi_J \Gamma^\ell \xi_I) - \frac{1}{8} \Gamma_{\ell m} \xi_K \epsilon^{JK} (\xi_J \Gamma^{\ell m} \xi_I) \ , \\ \Gamma_n \xi_I \cdot v^n &= \Gamma_n \xi_I \epsilon^{JK} (\xi_J \Gamma^n \xi_K) \\ &= \frac{1}{4} \xi_K \epsilon^{JK} (\xi_J \Gamma^n \Gamma_n \xi_I) + \frac{1}{4} \Gamma_\ell \xi_K \epsilon^{JK} (\xi_J \Gamma^n \Gamma^\ell \Gamma_n \xi_I) - \frac{1}{8} \Gamma_{\ell m} \xi_K \epsilon^{JK} (\xi_J \Gamma^n \Gamma^{\ell m} \Gamma_n \xi_I) \\ &= \frac{5}{4} \xi_K \epsilon^{JK} (\xi_J \xi_I) - \frac{3}{4} \Gamma_\ell \xi_K \epsilon^{JK} (\xi_J \Gamma^\ell \xi_I) - \frac{1}{8} \Gamma_{\ell m} \xi_K \epsilon^{JK} (\xi_J \Gamma^{\ell m} \xi_I) \ . \end{aligned} \quad (3.13)$$

Using

$$\xi_J \xi_I = -\frac{1}{2} \epsilon_{JI} \ , \quad \xi_J \Gamma^n \xi_I = -\frac{1}{2} \epsilon_{JI} v^n \ , \quad (3.14)$$

one finds

$$\begin{aligned} \xi_I &= \frac{1}{8} \xi_I + \frac{1}{8} \Gamma_\ell \xi_I \cdot v^\ell - \frac{1}{8} \Gamma_{\ell m} \xi_K \epsilon^{JK} (\xi_J \Gamma^{\ell m} \xi_I) \ , \\ \Gamma_n \xi_I \cdot v^n &= \frac{5}{8} \xi_I - \frac{3}{8} \Gamma_\ell \xi_I \cdot v^\ell - \frac{1}{8} \Gamma_{\ell m} \xi_K \epsilon^{JK} (\xi_J \Gamma^{\ell m} \xi_I) \ . \end{aligned} \quad (3.15)$$

By taking the difference between the above two equations, one finds the desired equality in (3.12).

Next, we show that $\delta_\xi \lambda_I = 0$ follows from the saddle point condition in (3.11). Recall

$$\delta_\xi \lambda_I = -\frac{1}{2} \Gamma^{mn} \xi_I F_{mn} + \Gamma^m \xi_I D_m \sigma + \xi_J (D_{KI} + 2\sigma t_{KI}) \epsilon^{JK} \ . \quad (3.16)$$

Assuming (3.11), all the terms on the right hand side except for the first one vanish. To show that the first term also vanishes, we notice

$$\begin{aligned} \Gamma^{mn} \xi_I F_{mn} &= \frac{1}{2} \Gamma^{mn} \xi_I \epsilon_{mnpqr} F^{pq} v^r = -\Gamma^{pqr} \xi_I F_{pq} v_r \\ &= -(\Gamma^{pq} \Gamma^r - \Gamma^p g^{qr} + \Gamma^q g^{pr}) \xi_I F_{pq} v_r = -\Gamma^{pq} \xi_I F_{pq} \ . \end{aligned} \quad (3.17)$$

Therefore, $\Gamma^{mn} \xi_I F_{mn}$ vanishes. Note that, in the second equality, we use $\Gamma^{12345} = 1$ leading to $\Gamma^{mn} \epsilon_{mnpqr} = -2\Gamma_{pqr}$. We also use (3.12) in the fourth equality. Since $\delta_\xi I|_{bos}$ vanishes by construction if $\delta_\xi \lambda_I = 0$, it follows that (3.11) and $\delta_\xi \lambda_I = 0$ are completely equivalent.

⁸Note the sign difference from the previous formula due to the fact that we are here dealing with Grassmann even spinors.

Let us now consider the saddle point equation in (3.11). We recall that $v^m F_{mn}$ means a translation (Lie derivative) with v^m and a gauge transformation with $v^m A_m$ of A_n . Thus, if we can take $v^m A_m = 0$ gauge, the condition $v^m F_{mn} = 0$ means A_n is constant in the v^m direction. Accordingly, we can think of the gauge field as being only on \mathbb{CP}^2 . It should be an instanton solution which follows from the condition $F_{mn} = \frac{1}{2}\epsilon_{mnpqr}v^p F^{qr}$. If, for example, a Wilson line for $v^m A_m$ does not vanish, we can not take the gauge. In this case, the saddle points are a combination of the Wilson line and the instantons. Therefore, we conclude that the path-integral is reduced to an integration over a generalization of instantons on \mathbb{CP}^2 and the covariantly constant σ on it. Needless to say, it is important to carry out explicitly this integral with the one loop determinant factor and saddle point action. We hope to return to this problem in the near future.

Hypermultiplets. Finally, we consider the localization of the hypermultiplets. If

$$\delta\psi = \epsilon^{IJ}\Gamma^m\xi_I D_m q_J + i\epsilon^{IJ}\xi_I(\sigma + m)q_J - 3t^{IJ}\xi_I q_J + \epsilon^{I'J'}\check{\xi}_{I'}F_{J'} \ , \quad (3.18)$$

then its complex conjugate should be (before rotating the integration contours for some variables),

$$(\delta\psi)^* = \Omega C \left(\epsilon^{IJ}\Gamma^m\xi_I D_m q_J + i\epsilon^{IJ}\xi_I(\sigma + m)q_J - 3t^{IJ}\xi_I q_J + \epsilon^{I'J'}\check{\xi}_{I'}F_{J'} \right) \ . \quad (3.19)$$

For positivity of the action of the hypermultiplets with the mass term, we have assumed that F is “pure imaginary”, q is “real” and the complex conjugate of m is the same as the one for σ . With the rotation of the contours for $\sigma, D_{IJ}, F_{J'}$ (and m) taken into account, this is modified to

$$(\delta\psi)^* = \Omega C \left(\epsilon^{IJ}\Gamma^m\xi_I D_m q_J - i\epsilon^{IJ}\xi_I(\sigma + m)q_J - 3t^{IJ}\xi_I q_J - \epsilon^{I'J'}\check{\xi}_{I'}F_{J'} \right) \ . \quad (3.20)$$

By taking its transpose one finds

$$(\delta\psi)^\dagger = \epsilon^{IJ}\xi_I C \Gamma^m D_m q_J \Omega + i\epsilon^{IJ}\xi_I C q_J \Omega(\sigma + m) - 3t^{IJ}\xi_I C q_J \Omega - \epsilon^{I'J'}\check{\xi}_{I'} C F_{J'} \Omega \ . \quad (3.21)$$

The regulator Lagrangian for the localization will be δV_{hyper} where

$$V_{hyper} = (\delta\psi)^\dagger \psi \ . \quad (3.22)$$

Then, the bosonic part of the regulator Lagrangian is $\delta V_{hyper}|_{bos} = (\delta\psi)^\dagger \delta\psi$ which becomes

$$\begin{aligned} \delta V_{hyper} = & \frac{1}{2}\epsilon^{IJ}D_m \bar{q}_I D_m q_J + 3v^m t^{IJ}\bar{q}_I D_m q_J + \frac{9}{4}t^{IJ}t_{IJ}\epsilon^{KL}\bar{q}_K q_L \\ & + w_{IJ}^{mn}D_m \bar{q}^I D_n q^J - \frac{1}{2}\epsilon^{IJ}\bar{q}_I(\sigma + m)^2 q_J - \frac{1}{2}\epsilon^{I'J'}\bar{F}_{I'}F_{J'} \ . \end{aligned} \quad (3.23)$$

Above, we have defined

$$w_{IJ}^{mn} \equiv \xi_I \Gamma^{mn} \xi_J \ , \quad (3.24)$$

which satisfies $w_{IJ}^{mn} = w_{JI}^{mn} = -w_{IJ}^{nm}$. Using the identities, we show from the Fierz identities that

$$\begin{aligned} \delta V_{hyper} = & \frac{1}{2}\epsilon^{IJ} \left(v^m D_m \bar{q}_I - 3t_I^K \bar{q}_K \right) \left(v^n D_n q_J - 3t_J^L q_L \right) \\ & + \frac{1}{8}\epsilon^{KL} \left(D^p \bar{q}_K - v^p(v^q D_q \bar{q}_K) + 2w_{K}^{pm} D_m \bar{q}_I \right) \left(D_p q_L - v_p(v^r D_r q_L) + 2w_{pL}^{Jn} D_n q_J \right) \\ & - \frac{1}{2}\epsilon^{IJ}\bar{q}_I(\sigma + m)^2 q_J - \frac{1}{2}\epsilon^{I'J'}\bar{F}_{I'}F_{J'} \ , \end{aligned} \quad (3.25)$$

where each term is positive definite for our choice of the contour. Therefore, the conditions for the saddle points are

$$\begin{aligned} C_K &\equiv v^m D_m q_K - 3t_K^L q_L = 0 , \\ C'_{pK} &\equiv D_p q_K - v_p (v^r D_r q_K) + 2w_{pK}^J D^n q_J = 0 , \\ (\sigma + m)q_J &= 0, \quad F_{J'} = 0 . \end{aligned} \quad (3.26)$$

Furthermore, in the Coulomb branch, where $(\sigma + m)$ does not have zero eigenvalue, the conditions are trivial; $q_J = 0$ and $F_{J'} = 0$. On the other hand, a Higgs or mixed branch would exist if $(\sigma + m)$ has zero eigenvalues.⁹

Let us derive some identities for w_{IJ}^{mn} . Multiplying $\xi_L \Gamma^{p_1 p_2 \dots}$ with the Fierz identity, we have the following:

$$0 = -w_{mnIJ} w_{KL}^{mn} + \epsilon_{IJ} \epsilon_{KL} + 2\epsilon_{IL} \epsilon_{JK} , \quad (3.27)$$

$$\begin{aligned} 0 &= 2(\epsilon_{IJ} \epsilon_{KL} + 2\epsilon_{IL} \epsilon_{JK}) v^p + 2v_m (\epsilon_{IJ} w_{KL}^{pm} - \epsilon_{KL} w_{IJ}^{pm}) \\ &\quad + \epsilon^{pmnqr} w_{mnIJ} w_{qrKL} , \end{aligned} \quad (3.28)$$

$$\begin{aligned} 0 &= 8\epsilon_{JK} w_{LI}^{pq} - 2\epsilon_{JI} w_{LK}^{pq} - 2\epsilon_{LK} w_{JI}^{pq} \\ &\quad + \epsilon_{JI} v_m \epsilon^{pqmrs} w_{rsLK} + \epsilon_{LK} v_m \epsilon^{pqmrs} w_{rsJI} \\ &\quad - 4(w_{JI}^{qn} w_{nLK}^p - w_{JI}^{pn} w_{nLK}^q) . \end{aligned} \quad (3.29)$$

Then, by applying ϵ^{IJ} to the identities (3.28) and (3.29), we obtain

$$0 = v_m w_{KL}^{pm} , \quad (3.30)$$

$$0 = 2w_{IJ}^{pq} + v_m \epsilon^{pqrs} w_{rsIJ} . \quad (3.31)$$

In addition, by applying ϵ^{LI} to the identities (3.27), (3.28) and (3.29), we obtain

$$0 = w_{KI}^{mn} w_{mnJ}^I + 3\epsilon_{JK} , \quad (3.32)$$

$$0 = 6\epsilon_{JK} v^p - \epsilon^{pmnqr} w_{mnIJ} w_{qrK}^I , \quad (3.33)$$

$$0 = -2w_{JK}^{qp} + \epsilon^{IL} (w_{IJ}^{qn} w_{nLK}^p + w_{IK}^{qn} w_{nLJ}^p) . \quad (3.34)$$

By taking the square of the first term and the second term of (3.31), we have the following identity

$$\epsilon^{JL} (w_{IJ}^{pm} w_{mKL}^q + w_{IJ}^{qm} w_{mKL}^p) = \frac{3}{2} \epsilon_{IK} (g^{pq} - v^p v^q) . \quad (3.35)$$

With this and (3.34) we find

$$\epsilon^{KL} w_{IK}^{pm} w_{mJL}^q = w_{IJ}^{pq} + \frac{3}{4} (g^{pq} - v^p v^q) \epsilon_{IJ} . \quad (3.36)$$

Using the identities we derived, we can show the form of the bosonic part in (3.25) where the positive definiteness is manifest.

We can also show that $\delta\psi = 0$ is indeed satisfied by the saddle point condition. It is easy to see that $\xi_K \delta\psi = 0$ and $\xi_K \Gamma^p \delta\psi = 0$ on the saddle points. We also find the identity: $\xi_K \Gamma^{pq} \delta\psi + v^q (\xi_K \Gamma^p \delta\psi) - v^p (\xi_K \Gamma^q \delta\psi) - 2(\xi_I \delta\psi) w_{KJ}^{pq} \epsilon^{IJ} = 0$. Therefore, we have indeed shown that $\delta\psi = 0$ is indeed satisfied on the saddle points.

⁹However, it is possible that there are no solutions of $C_K = C'_{pK} = 0$ on S^5 .

4. Conclusion

In this paper, we have constructed 5d SUSY gauge theories on the five-sphere with vector and hypermultiplets. We have shown that with the localization terms the path-integral can be restricted to an integration over a generalization of instantons on \mathbb{CP}^2 and the covariantly constant Coulomb moduli. It is interesting that instantons in 4d appear in 5d SUSY gauge theories on S^5 . If we regard the 5d theory as a compactification of a 6d conformal field theory, the 4d instanton in the former can be interpreted as a Kaluza-Klein particle of the latter. Then, the choice of the Killing spinor, which determines v^m , corresponds to a choice of a Wick rotated time direction.

It is of great interest to study further the appearance of instantons on \mathbb{CP}^2 in the context of 5d SUSY theories on S^5 . Evaluating the integration over the moduli space of instantons and hence finding the localized path-integral are key steps to take in future work. In addition, of great interest is the study of SUSY gauge theories on deformed S^5 in line with the work in [23] for the case of the squashed three-sphere.

Acknowledgements

K. H. would like to thank the Perimeter Institute for hospitality. R.-K. S. would like to thank the Yukawa Institute at Kyoto University for hospitality, and the Global COE program for support. S. T. would like to thank K. Sakai and M. Taki for helpful discussions. The work of S. T. is partly supported by the Japan Ministry of Education, Culture, Sports, Science and Technology (MEXT), and by the Grant-in-Aid for the Global COE program “The Next Generation of Physics, Spun from Universality and Emergence” from the MEXT.

Note added:

As this article neared completion, we became aware of the preprint [24] in which topological SUSY gauge theories on 5d manifolds with circle fibration structure, including spheres, are constructed following the 3d case [25]. Our 5d SUSY gauge theories on S^5 with the terms needed for localization would coincide with their topological one if we ignore the original action. This would be equivalent to taking the (formal) strong coupling limit.

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